arXiv:math/9901017v1 [math.GN] 5 Jan 1999

Idealization of Ganster–Reilly decomposition theorems*

Julian Dontchev
Department of Mathematics
University of Helsinki
PL 4, Yliopistonkatu 5
00014 Helsinki
Finland

Abstract

In 1990, Ganster and Reilly [6] proved that a function $f\colon (X,\tau) \rightarrow (Y,\sigma)$ is continuous if and only if it is precontinuous and LC-continuous. In this paper we extend their decomposition of continuity in terms of ideals. We show that a function $f\colon (X,\tau,\mathcal{I}) \rightarrow (Y,\sigma)$ is continuous if and only if it is $\mathcal{I}$-continuous and $\mathcal{I}$-LC-continuous. We also provide a decomposition of $\mathcal{I}$-continuity.

1 Introduction to topological ideals

In [6, 7, 8], Ganster and Reilly gave several new decompositions of continuity.

Let A be a subset of a topological space (X,τ). Following Kronheimer [12], we call the interior of the closure of A, denoted by A^+, the *consolidation* of A. Sets included in their consolidation are called *preopen* or *locally dense* [3]. If A is the intersection of an open and a closed (resp. regular closed) set, then A is called *locally closed* (resp. $\mathcal{A}$*-set* [18]). A function $f\colon (X,\tau) \rightarrow (Y,\sigma)$ is called *precontinuous* (resp. LC*-continuous* [5], $\mathcal{A}$*-continuous* [18]) if the preimage of every open set is preopen (resp. locally closed, $\mathcal{A}$-set). The following theorem is due to Ganster and Reilly [6, Theorem 4 (iv) and (v)].

*1991 Math. Subject Classification — Primary: 54C08, 54A20; Secondary: 54A05, 54C10.
Key words and phrases — decomposition of continuity, topological ideal, $\mathcal{I}$-open set, preopen, locally closed, pre-$\mathcal{I}$-open set, $\mathcal{I}$-continuous function.
Research supported partially by the Ella and Georg Ehrnrooth Foundation at Merita Bank, Finland.

Theorem 1.1 [6] *For a function $f\colon (X,\tau) \to (Y,\sigma)$ the following conditions are equivalent:*

(1) *f is continuous.*

(2) *f is precontinuous and $\mathcal{A}$-continuous.*

(3) *f is precontinuous and LC-continuous.*

The aim of this paper is to present an idealized version of the Ganster–Reilly decomposition theorem.

A nonempty collection $\mathcal{I}$ of subsets on a topological space (X,τ) is called an *ideal* on X if it satisfies the following two conditions:

(1) If $A \in \mathcal{I}$ and $B \subseteq A$, then $B \in \mathcal{I}$ (heredity).

(2) If $A \in \mathcal{I}$ and $B \in \mathcal{I}$, then $A \cup B \in \mathcal{I}$ (finite additivity).

A *σ-ideal* on a topological space (X,τ) is an ideal which satisfies:

(3) If $\{A_i\colon i = 1,2,3,\ldots\} \subseteq \mathcal{I}$, then $\bigcup\{A_i\colon i = 1,2,3,\ldots\} \in \mathcal{I}$ (countable additivity).

If $X \notin \mathcal{I}$, then $\mathcal{I}$ is called a *proper* ideal. The collection of the complements of all elements of a proper ideal is a filter, hence proper ideals are sometimes called *dual filters*.

The following collections form important ideals on a topological space (X,τ): the ideal of all finite sets $\mathcal{F}$, the σ-ideal of all countable sets $\mathcal{C}$, the ideal of all closed and discrete sets $\mathcal{CD}$, the ideal of all nowhere dense sets $\mathcal{N}$, the σ-ideal of all meager sets $\mathcal{M}$, the ideal of all scattered sets $\mathcal{S}$ (here X must be T_0) and the σ-ideal of all Lebesgue null sets $\mathcal{L}$ (here X is the real line).

An *ideal topological space* is a topological space (X,τ) with an ideal $\mathcal{I}$ on X and is denoted by $(X,\tau,\mathcal{I})$. For a subset $A \subseteq X$, $A^*(\mathcal{I}) = \{x \in X\colon \text{for every } U \in \tau(x), U \cap A \notin \mathcal{I}\}$ is called the *local function* of A with respect to $\mathcal{I}$ and τ [10, 13]. We simply write A^* instead of $A^*(\mathcal{I})$ in case there is no chance for confusion. Note that often X^* is a proper subset of X. The hypothesis $X = X^*$ was used by Hayashi in [9], while the hypothesis $\tau \cap \mathcal{I} = \emptyset$ was used by Samuels in [17]. In fact, those two conditions are equivalent [10, Theorem 6.1] and we call the ideal topological spaces which satisfy this hypothesis *Hayashi-Samuels spaces*. Note that $(X,\tau,\{\emptyset\})$ and $(X,\tau,\mathcal{N})$ are always Hayashi-Samuels spaces; also $(\mathbb{R},\tau,\mathcal{F})$ is a Hayashi-Samuels space, where τ denotes the usual topology on the real line $\mathbb{R}$.

For every ideal topological space $(X, \tau, \mathcal{I})$, there exists a topology $\tau^*(\mathcal{I})$, finer than τ, generated by the base $\beta(\mathcal{I}, \tau) = \{U \setminus I : U \in \tau \text{ and } I \in \mathcal{I}\}$. In general, $\beta(\mathcal{I}, \tau)$ is not always a topology [10]. When there is no chance for confusion, $\tau^*(\mathcal{I})$ is denoted by τ^*. Observe additionally that $\mathrm{Cl}^*(A) = A \cup A^*$ defines a Kuratowski closure operator for (the same topology) $\tau^*(\mathcal{I})$.

Recall that $A \subseteq (X, \tau, \mathcal{I})$ is called *$\star$-dense-in-itself* [9] (resp. *τ^*-closed* [10], *$\star$-perfect* [9]) if $A \subseteq A^*$ (resp. $A^* \subseteq A$, $A = A^*$).

It is interesting to note that $A^*(\mathcal{I})$ is a generalization of closure points, ω-accumulation points and condensation points. Recall that the set of all ω-accumulation points of subset A of a topological space (X, τ) is $A^\omega = \{x \in X : U \cap A \text{ is infinite for every } U \in \mathcal{N}(x)\}$. The set of all condensation points of A is $A^k = \{x \in X : U \cap A \text{ is uncountable for every } U \in \mathcal{N}(x)\}$. It is easily seen that $\mathrm{Cl}(A) = A^*(\{\emptyset\})$, $A^\omega = A^*(\mathcal{F})$ and $A^k = A^*(\mathcal{C})$. Note here that in T_1-spaces the concepts of ω-accumulation points and limit points coincide.

In 1990, D. Janković and T.R. Hamlett introduced the notion of $\mathcal{I}$-open sets in ideal topological spaces. Given an ideal topological space $(X, \tau, \mathcal{I})$ and $A \subseteq X$, A is said to be *$\mathcal{I}$-open* [11] if $A \subseteq \mathrm{Int}(A^*)$. We denote by $IO(X, \tau, \mathcal{I}) = \{A \subseteq X : A \subseteq \mathrm{Int}(A^*)\}$ or simply write $IO(X, \tau)$ or $IO(X)$ when there is no chance for confusion with the ideal. A subset $F \subseteq (X, \tau, \mathcal{I})$ is called [1] *$\mathcal{I}$-closed* if its complement is $\mathcal{I}$-open. Note that X need not be an $\mathcal{I}$-open subset. Thus, not only are $\mathcal{I}$-open and τ^*-open sets are different concepts, but the former do not give a topology. In the extreme case when $\mathcal{I}$ is the maximal ideal of all subsets of X, only the void subset is $\mathcal{I}$-open.

A function $f : (X, \tau, \mathcal{J}_1) \to (Y, \sigma, \mathcal{J}_2)$ is said to be *$\mathcal{I}$-continuous* (resp. *$\mathcal{I}$-open*, *$\mathcal{I}$-closed*) if for every $V \in \sigma$ (resp. $U \in \tau$, U closed in X), $f^{-1}(V) \in IO(X, \tau)$ (resp. $f(U) \in IO(X, \tau)$, $f(U)$ is $\mathcal{I}$-closed). The definitions are due to Monsef *et al.* [1].

In [15], a topology τ^α has been introduced by defining its open sets to be the α-sets, that is the sets $A \subseteq X$ with $A \subset \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(A)))$. Observe that $\tau^\alpha = \tau^*(\mathcal{N})$.

2 Pre-$\mathcal{I}$-open sets

Definition 1 A subset of an ideal topological space $(X, \tau, \mathcal{I})$ is called *pre-$\mathcal{I}$-open* if $A \subseteq \text{Int}(\text{Cl}^*(A))$.

We denote by $PIO(X, \tau, \mathcal{I})$ the family of all pre-$\mathcal{I}$-open subsets of $(X, \tau, \mathcal{I})$ or simply write $PIO(X, \tau)$ or $PIO(X)$ when there is no chance for confusion with the ideal. We call a subset $A \subseteq (X, \tau, \mathcal{I})$ *pre-$\mathcal{I}$-closed* if its complement is pre-$\mathcal{I}$-open.

Although $\mathcal{I}$-openness and openness are independent concepts [1, Examples 2.1 and 2.2], pre-$\mathcal{I}$-openness is related to both of them as the following two results show.

Proposition 2.1 *Every $\mathcal{I}$-open set is pre-$\mathcal{I}$-open.*

Proof. Let $(X, \tau, \mathcal{I})$ be an ideal topological space and let $A \subseteq X$ be $\mathcal{I}$-open. Then $A \subseteq \text{Int}(A^*) \subseteq \text{Int}(A^* \cup A) = \text{Int}(\text{Cl}^*(A))$. $\square$

Proposition 2.2 *Every open set is pre-$\mathcal{I}$-open.*

Proof. Let $A \subseteq (X, \tau, \mathcal{I})$ be open. Then $A \subseteq \text{Int}A \subseteq \text{Int}(A^* \cup A) = \text{Int}(\text{Cl}^*(A))$. $\square$

The converse in the proposition above is not necessarily true as shown by the following two examples.

Example 2.3 A pre-$\mathcal{I}$-open set, even an open set, need not be $\mathcal{I}$-open. Let $X = \{a, b, c, d\}$, $\tau = \{\emptyset, \{a, c\}, \{d\}, \{a, c, d\}, X\}$ and $\mathcal{I} = \{\emptyset, \{c\}, \{d\}, \{c, d\}\}$. Set $A = \{a, c, d\}$. Then $A \in \tau$ and hence $A \in PIO(X)$ but $A \notin IO(X)$ [1, Example 2.2].

Example 2.4 Let (X, τ) be the real line with the usual topology and let $\mathcal{F}$ be as mentioned before the ideal of all finite subsets of X. Let $\mathbb{Q}$ be the set of all rationals. Since $\mathbb{Q}^*(\mathcal{F}) = X$, then $\mathbb{Q}$ is pre-$\mathcal{I}$-open (even $\mathcal{I}$-open). But clearly $\mathbb{Q} \notin \tau$.

Our next two results together with Propsoition 2.1 and Propsoition 2.2 shows that the class of pre-$\mathcal{I}$-open sets is properly placed between the classes of $\mathcal{I}$-open and preopen sets as well as between the classes of open and preopen sets.

Proposition 2.5 *Every pre-$\mathcal{I}$-open set is preopen.*

Proof. Let $(X, \tau, \mathcal{I})$ be an ideal topological space and let $A \in PIO(X)$. Then $A \subseteq \mathrm{Int}(\mathrm{Cl}^*(A)) = \mathrm{Int}(A^* \cup A) \subseteq \mathrm{Int}(\mathrm{Cl}(A) \cup A) = \mathrm{Int}(\mathrm{Cl}(A))$. □

Example 2.6 A preopen set need not be pre-$\mathcal{I}$-open. Every singleton (for example) in an indiscrete topological space with cardinality at least two is preopen but if we set $\mathcal{I}$ to be the maximal ideal, i.e., $\mathcal{I} = \mathcal{P}(X)$, then it is easy to see that none of the singletons is pre-$\mathcal{I}$-open.

Proposition 2.7 *For an ideal topological space $(X, \tau, \mathcal{I})$ and $A \subseteq X$ we have:*

(i) *If $\mathcal{I} = \emptyset$, then A is pre-$\mathcal{I}$-open if and only if A is preopen.*

(ii) *If $\mathcal{I} = \mathcal{P}(X)$, then A is pre-$\mathcal{I}$-open if and only if $A \in \tau$.*

(iii) *If $\mathcal{I} = \mathcal{N}$, then A is pre-$\mathcal{I}$-open if and only if A is preopen.*

Proof. (i) Necessity is shown in Proposition 2.5. For sufficiency note that in the case of the minimal ideal $A^* = \mathrm{Cl}(A)$.

(ii) Necessity: If $A \in PIO(X)$, then $A \subseteq \mathrm{Int}(\mathrm{Cl}^*(A)) = \mathrm{Int}(A \cup A^*) = \mathrm{Int}(A \cup \emptyset) = \mathrm{Int}A$. Sufficiency is given in Proposition 2.2.

(iii) By Proposition 2.5 we need to show only sufficiency. Note that the local function of A with respect to $\mathcal{N}$ and τ can be given explicitly [19]. We have:

$$A^*(\mathcal{N}) = \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(A))).$$

Thus A is pre-$\mathcal{I}$-open if and only if $A \subseteq \mathrm{Int}(A \cup \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(A))))$. Assume that A is preopen. Since always $\mathrm{Int}(\mathrm{Cl}(A)) \subseteq A \cup \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(A)))$, then $A \subseteq \mathrm{Int}(A \cup \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(A)))) = \mathrm{Int}(A \cup A^*(\mathcal{N})) = \mathrm{Int}(\mathrm{Cl}^*(A))$ or equivalently A is pre-$\mathcal{I}$-open. □

The intersection of even two pre-$\mathcal{I}$-open sets need not be pre-$\mathcal{I}$-open as shown in the following example.

Example 2.8 Let $X = \{a, b, c\}$, $\tau = \{\emptyset, \{a, b\}, X\}$ and $\mathcal{I} = \{\emptyset, \{c\}\}$. Set $A = \{a, c\}$ and $B = \{b, c\}$. Since $A^* = B^* = X$, then both A and B are pre-$\mathcal{I}$-open. But on the other hand $A \cap B = \{c\} \notin PIO(X)$.

Lemma 2.9 [10, Theorem 2.3 (g)] *Let* $(X, \tau, \mathcal{I})$ *be an ideal topological space and let* $A \subseteq X$. *Then* $U \in \tau \Rightarrow U \cap A^* = U \cap (U \cap A)^* \subseteq (U \cap A)^*$. $\Box$

Proposition 2.10 *Let* $(X, \tau, \mathcal{I})$ *be an ideal topological space with* $\triangle$ *an arbitrary index set. Then:*

(i) *If* $\{A_\alpha : \alpha \in \triangle\} \subseteq PIO(X)$, *then* $\cup\{A_\alpha : \alpha \in \triangle\} \in PIO(X)$.

(ii) *If* $A \in PIO(X)$ *and* $U \in \tau$, *then* $A \cap U \in PIO(X)$.

(iii) *If* $A \in PIO(X)$ *and* $B \in \tau^\alpha$, *then* $A \cap B \in PO(X)$.

(iv) *If* $A \in PIO(X)$ *and* $B \in SO(X)$, *then* $A \cap B \in SO(A)$.

(v) *If* $A \in PIO(X)$ *and* $B \in SO(X)$, *then* $A \cap B \in PO(B)$.

Proof. (i) Since $\{A_\alpha : \alpha \in \triangle\} \subseteq PIO(X)$, then $A_\alpha \subseteq \mathrm{Int}(\mathrm{Cl}^*(A_\alpha))$ for every $\alpha \in \triangle$. Thus $\cup_{\alpha\in\triangle} A_\alpha \subseteq \cup_{\alpha\in\triangle} \mathrm{Int}(\mathrm{Cl}^*(A_\alpha)) \subseteq \mathrm{Int}(\cup_{\alpha\in\triangle} \mathrm{Cl}^*(A_\alpha)) = \mathrm{Int}(\cup_{\alpha\in\triangle}(A_\alpha^* \cup A_\alpha)) = \mathrm{Int}((\cup_{\alpha\in\triangle} A_\alpha^*) \cup (\cup_{\alpha\in\triangle} A_\alpha)) \subseteq \mathrm{Int}((\cup_{\alpha\in\triangle} A_\alpha)^* \cup (\cup_{\alpha\in\triangle} A_\alpha)) = \mathrm{Int}(\mathrm{Cl}^*(\cup_{\alpha\in\triangle} A_\alpha))$.

(ii) By assumption $A \subseteq \mathrm{Int}(\mathrm{Cl}^*(A))$ and $U \subseteq \mathrm{Int}(U)$. Thus applying Lemma 2.9, $A \cap U \subseteq \mathrm{Int}(\mathrm{Cl}^*(A)) \cap \mathrm{Int}(U) \subset \mathrm{Int}(\mathrm{Cl}^*(A) \cap U) = \mathrm{Int}((A^* \cup A) \cap U) = \mathrm{Int}((A^* \cap U) \cup (A \cap U)) \subseteq \mathrm{Int}((A \cap U)^* \cup (A \cap U)) = \mathrm{Int}(\mathrm{Cl}^*(A \cap U))$.

(iii) Since the intersection of a preopen set and an α-set is always a preopen set, then the claim is clear due to Propsoition 2.5.

(iv) and (v) It is proved in [16] that the intersection of a preopen and a semi-open set is a preopen subset of the semi-open set and a semi-open subset of the preopen set. Thus the claim follows from Proposition 2.5. $\Box$

Corollary 2.11 (i) *The intersection of an arbitrary family of pre-$\mathcal{I}$-closed sets is a pre-$\mathcal{I}$-closed set.*

(ii) *The union of a pre-$\mathcal{I}$-closed set and a closed set is pre-$\mathcal{I}$-closed.* $\Box$

Recall that (X, τ) is called *submaximal* if every dense subset of X is open.

Lemma 2.12 [14, Lemma 5] *If* (X, τ) *is submaximal, then* $PO(X, \tau) = \tau$. $\Box$

Corollary 2.13 *If (X, τ) is submaximal, then for any ideal $\mathcal{I}$ on X, $\tau = PIO(X)$.* $\square$

Remark 2.14 By Proposition 2.10, the intersection of a pre-$\mathcal{I}$-open set and an open set is pre-$\mathcal{I}$-open. However, the intersection of a pre-$\mathcal{I}$-open set and an $\mathcal{I}$-open set is not necessarily pre-$\mathcal{I}$-open, since in Example 2.8 $\{c\} = A \cap B$ is not pre-$\mathcal{I}$-open, although A is pre-$\mathcal{I}$-open (even $\mathcal{I}$-open) and B is $\mathcal{I}$-open.

Remark 2.15 (i) In an ideal topological space $(X, \tau, \mathcal{I})$, the subset X need not always be $\mathcal{I}$-open. However, X is always pre-$\mathcal{I}$-open.

(ii) If $A \subseteq (X, \tau, \mathcal{I})$ is $\star$-perfect, then $A \in \tau$ if and only if $A \in IO(X)$ if and only if $A \in PIO(X)$.

Problem. The class of ideal topological spaces $(X, \tau, \mathcal{I})$ with $PIO(X, \tau, \mathcal{I}) \subseteq \tau^*(\mathcal{I})$ is probably of some interest. Call these spaces *$\mathcal{I}$-strongly irresolvable*. It is not difficult to observe that in the trivial case $\mathcal{I} = \{\emptyset\}$, we have the class of strongly irresolvable spaces which were introduced in 1991 by Foran and Liebnitz [4]. Note also that in the case of the maximal ideal $\mathcal{P}(X)$, every ideal topological space is $\mathcal{P}(X)$-strongly irresolvable. It is the author's belief that further study of $\mathcal{I}$-strongly irresolvable spaces is worthwhile.

3 A decomposition of $\mathcal{I}$-continuity

Definition 2 A function $f\colon (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is called *pre-$\mathcal{I}$-continuous* if for every $V \in \sigma$, $f^{-1}(V) \in PIO(X, \tau)$.

In the notion of Proposition 2.2 we have the following result:

Proposition 3.1 *Every continuous function $f\colon (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is pre-$\mathcal{I}$-continuous.* $\square$

The converse is not true in general as shown in the following example.

Example 3.2 Consider first the classical Dirichlet function $f\colon \mathbb{R} \to \mathbb{R}$:

$$f(x) = \begin{cases} 1, & x \in \mathbb{Q}, \\ 0, & \text{otherwise.} \end{cases}$$

Let $\mathcal{F}$ be the ideal of all finite subsets of $\mathbb{R}$. The Dirichlet function $f\colon (\mathbb{R}, \tau, \mathcal{F}) \to (\mathbb{R}, \tau)$ is pre-$\mathcal{I}$-continuous, since every point of $\mathbb{R}$ belongs to the local function of the rationals with respect to $\mathcal{F}$ and τ as well as to the local function of the irrationals. Hence f is even $\mathcal{I}$-continuous. But on the other hand the Dirichlet function is not continuous at any point of its domain.

Due to Proposition 2.1 we have the next result:

Proposition 3.3 *Every $\mathcal{I}$-continuous function $f\colon (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is pre-$\mathcal{I}$-continuous.* □

The reverse is again not true as the following example shows.

Example 3.4 Let $(X, \tau, \mathcal{I})$ be the space from Example 2.3 and let $\sigma = \{\emptyset, \{a, c, d\}, X\}$. Then the identity function $f\colon (X, \tau, \mathcal{I}) \to (X, \sigma)$ is pre-$\mathcal{I}$-continuous but not $\mathcal{I}$-continuous.

From Proposition 2.5 we have:

Proposition 3.5 *Every pre-$\mathcal{I}$-continuous function $f\colon (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is precontinuous.* □

Example 3.6 A precontinuous function need not be pre-$\mathcal{I}$-continuous. Let (X, τ) be the real line with the indiscrete topology and (Y, σ) the real line with the usual topology. The identity function $f\colon (X, \tau, \mathcal{P}(X)) \to (Y, \sigma)$ is precontinuous but not pre-$\mathcal{I}$-continuous.

Proposition 3.7 *For a function $f\colon (X, \tau, \mathcal{I}) \to (Y, \sigma)$ the following conditions are equivalent:*

(1) *f is pre-$\mathcal{I}$-continuous.*

(2) *For each $x \in X$ and each $V \in \sigma$ containing $f(x)$, there exists $W \in PIO(X)$ containing x such that $f(W) \subseteq V$.*

(3) *For each $x \in X$ and each $V \in \sigma$ containing $f(x)$, $\mathrm{Cl}^*(f^{-1}(V))$ is a neighborhood of x.*

(4) *The inverse image of each closed set in (Y, σ) is pre-$\mathcal{I}$-closed.*

Proof. (1) $\Rightarrow$ (2) Let $x \in X$ and let $V \in \sigma$ such that $f(x) \in V$. Set $W = f^{-1}(V)$. By (1), W is pre-$\mathcal{I}$-open and clearly $x \in W$ and $f(W) \subseteq V$.

(2) $\Rightarrow$ (3) Since $V \in \sigma$ and $f(x) \in V$, then by (2) there exists $W \in PIO(X)$ containing x such that $f(W) \subseteq V$. Thus, $x \in W \subseteq \text{Int}(\text{Cl}^*(W)) \subseteq \text{Int}(\text{Cl}^*(f^{-1}(V))) \subseteq \text{Cl}^*(f^{-1}(V))$. Hence, $\text{Cl}^*(f^{-1}(V))$ is a neighborhood of x.

(3) $\Rightarrow$ (1) and (1) $\Leftrightarrow$ (4) are obvious. $\square$

The composition of two pre-$\mathcal{I}$-continuous functions need not be always pre-$\mathcal{I}$-continuous as the following example shows.

Example 3.8 Let $\mathbb{R}$ be again the real line and τ the usual topology. Note that the identity function $g\colon (\mathbb{R}, \tau, \mathcal{P}(X)) \to (\mathbb{R}, \tau, \mathcal{F})$ is pre-$\mathcal{I}$-continuous and also the Dirichlet function $f\colon (\mathbb{R}, \tau, \mathcal{F}) \to (\mathbb{R}, \tau)$ is pre-$\mathcal{I}$-continuous (Example 3.2). But their composition $(f \circ g)\colon (\mathbb{R}, \tau, \mathcal{P}(X)) \to (\mathbb{R}, \sigma)$ is not pre-$\mathcal{I}$-continuous, since (for example) $f^{-1}\{(0,2)\} = \mathbb{Q} \notin PIO(\mathbb{R}, \tau, \mathcal{P}(X))$.

However the following result holds.

Proposition 3.9 *Let $f\colon (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$ and $g\colon (Y, \sigma, \mathcal{J}) \to (Z, \upsilon)$ be two functions, where $\mathcal{I}$ and $\mathcal{J}$ are ideals on X and Y respectively. Then:*

(i) *$g \circ f$ is pre-$\mathcal{I}$-continuous, if f is pre-$\mathcal{I}$-continuous and g is continuous.*

(ii) *$g \circ f$ is precontinuous, if g is continuous and f is pre-$\mathcal{I}$-continuous.*

Proof. Obvious. $\square$

Hayashi [9] defined a set A to be $\star$-dense-in-itself if $A \subseteq A^*(\mathcal{I})$. We say that a function $f\colon (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is *$\star$-$\mathcal{I}$-continuous* if the preimage of every open set in (Y, σ) is $\star$-dense-in-itself in $(X, \tau, \mathcal{I})$. In what follows, we try do decompose $\mathcal{I}$-continuity but before that we will give a decomposition of $\mathcal{I}$-openness. Our next two examples (the ones after Proposition 3.10 and Proposition 3.11) will show that pre-$\mathcal{I}$-continuity and $\star$-$\mathcal{I}$-continuity are independent concepts.

Proposition 3.10 *For a subset $A \subseteq (X, \tau, \mathcal{I})$ the following conditions are equivalent:*

(1) *A is $\mathcal{I}$-open.*

(2) *A is pre-$\mathcal{I}$-open and $\star$-dense-in-itself.*

Proof. (1) By Proposition 2.1, every $\mathcal{I}$-open set is pre-$\mathcal{I}$-open. On the other hand $A \subseteq \text{Int}(A^*) \subset A^*$, which shows that A is $\star$-dense-in-itself.

(2) $\Rightarrow$ (1) By assumption $A \subseteq \text{Int}(\text{Cl}^*(A)) = \text{Int}(A^* \cup A) = \text{Int}(A^*)$ or equivalently A is $\mathcal{I}$-open. $\square$

Thus we have the following decomposition of $\mathcal{I}$-continuity:

Theorem 3.11 *For a function $f\colon (X, \tau\mathcal{I}) \to (Y, \sigma)$ the following conditions are equivalent:*

(1) *f is $\mathcal{I}$-continuous.*

(2) *f is pre-$\mathcal{I}$-continuous and $\star$-$\mathcal{I}$-continuous.* $\square$

Example 3.12 The identity function $f\colon (\mathbb{R}, \tau, \mathcal{P}(X)) \to (\mathbb{R}, \tau)$, where τ stands for the usual topology on the real line is pre-$\mathcal{I}$-continuous as mentioned in Example 3.8 but not $\star$-$\mathcal{I}$-continuous, since the local function of every subset of $\mathbb{R}$ with respect to $\mathcal{P}(X)$ and τ coincides with the void set.

Example 3.13 Note that in the case of the minimal ideal every function is $\star$-$\mathcal{I}$-continuous, since the local function of every set coincides with its closure. But since not every function is precontinuous, then $\star$-$\mathcal{I}$-continuity does not always imply pre-$\mathcal{I}$-continuity.

Remark 3.14 Of course a very appropriate example would be the construction of a space with a fixed ideal on it and finding topologies on the space such that certain functions would show the independence of pre-$\mathcal{I}$-continuity and $\star$-$\mathcal{I}$-continuity as well as the fact that they are both weaker than $\mathcal{I}$-continuity. Such an example is the following: Let $X = \{a, b, c\}$, $\mathcal{I} = \{\emptyset, \{c\}\}$, $\tau = \{\emptyset, \{b\}, X\}$, $\sigma = \{\emptyset, \{c\}, X\}$, $\nu = \{\emptyset, \{a\}, X\}$. The identity function $f\colon (X, \tau, \mathcal{I}) \to (X, \nu, \mathcal{I})$ is $\star$-$\mathcal{I}$-continuous but neither $\mathcal{I}$-continuous nor pre-$\mathcal{I}$-continuous. On the other hand the identity function $g\colon (X, \sigma, \mathcal{I}) \to (X, \sigma, \mathcal{I})$ is pre-$\mathcal{I}$-continuous but neither $\mathcal{I}$-continuous nor $\star$-$\mathcal{I}$-continuous.

In the case when $\mathcal{N}$ is the ideal of all nowhere dense subsets precontinuity coincides with pre-$\mathcal{I}$-continuity, while β-continuity is equivalent to $\star$-$\mathcal{I}$-continuity due to Proposition 2.7. Recall that a function $f\colon (X,\tau) \to (Y,\sigma)$ is called *β-continuous* (or sometimes *semi-precontinuous*) if the preimage of every open set in (Y,σ) is β-open in (X,τ), where a set A is called β-open if $A \subseteq \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(A)))$. It is clear, since every preopen set is β-open but not vice versa, that the family of all pre-$\mathcal{I}$-open subsets of an ideal topological space $(X,\tau,\mathcal{I})$ is a proper subset of the family of all β-open sets.

Consider next the ideal of all meager subsets. Recall that a set is *meager* if it is a countable union of nowhere dense sets. Meager sets are called often sets of *first category*. If a set is not meager it is said to be of *second category*. The points of second category of A are the points of $A^*(\mathcal{M})$ [13]. In 1922 Blumberg [2] called a point x of a space (X,τ) *inexhaustibly approached* by $A \subseteq X$ if $x \in A^*(\mathcal{M})$. If we call the set A *inexhaustibly approached* when every point of A is inexhaustibly approached by A, then clearly a function is $\star$-$\mathcal{M}$-continuous if and only if the inverse image of every open set is inexhaustibly approached.

4 Idealized Ganster–Reilly decomposition theorem

A subset A of an ideal topological space $(X,\tau,\mathcal{I})$ is called *$\mathcal{I}$-locally closed* if $A = U \cap V$, where $U \in \tau$ is V is $\star$-perfect. Note that in the case of the minimal ideal, $\mathcal{I}$-locally closed is equivalent to locally closed, while $\mathcal{N}$-locally closed is equivalent to the Tong's notion of an $\mathcal{A}$-set from [18].

Proposition 4.1 *For a subset $A \subseteq (X,\tau,\mathcal{I})$ of a Hayashi-Samuels space the following conditions are equivalent:*

(1) *A is open.*

(2) *A is pre-$\mathcal{I}$-open and $\mathcal{I}$-LC-continuous.*

Proof. (1) $\Rightarrow$ (2) The first part is Proposition 2.2. For the second part, note that $A = A \cap X$, where $A \in \tau$ and X is $\star$-perfect.

(2) $\Rightarrow$ (1) By assumption $A \subseteq \mathrm{Int}(\mathrm{Cl}^*(A)) = \mathrm{Int}(\mathrm{Cl}^*(U \cap V))$, where $U \in \tau$ and V is $\star$-perfect. Hence, $A = U \cap A \subseteq U \cap (\mathrm{Int}(\mathrm{Cl}^*(U)) \cap \mathrm{Int}(\mathrm{Cl}^*(V))) = U \cap \mathrm{Int}(V \cup V^*) = \mathrm{Int}(U) \cap \mathrm{Int}(V) = \mathrm{Int}(U \cap V) = \mathrm{Int}(A)$. This is shows that $A \in \tau$. $\square$

Definition 3 A function $f\colon (X,\tau,\mathcal{I}) \to (Y,\sigma)$ is called *$\mathcal{I}$-LC-continuous* if for every $V \in \sigma$, $f^{-1}(V)$ is $\mathcal{I}$-LC-closed.

Proposition 4.2 *Let $(X,\tau,\mathcal{I})$ be a Hayashi-Samuels space. Then, every continuous function $f\colon (X,\tau,\mathcal{I}) \to (Y,\sigma)$ is $\mathcal{I}$-LC-continuous.* $\square$

The converse is not true in general, since in the case of the minial ideal $(X,\tau,\mathcal{I})$ is a Hayashi-Samuels space but (usual) LC-continuous functions need not be LC-continuous [5].

Now, in the notion of Proposition 4.1, we have the following idealized decomposition of continuity:

Theorem 4.3 *Let $(X,\tau,\mathcal{I})$ be a Hayashi-Samuels space. For a function $f\colon (X,\tau,\mathcal{I}) \to (Y,\sigma)$ the following conditions are equivalent:*

(1) *f is continuous.*

(2) *f is pre-$\mathcal{I}$-continuous and $\mathcal{I}$-LC-continuous.* $\square$

Remark 4.4 From the particular cases $\mathcal{I} = \{\emptyset\}$ and $\mathcal{I} = \mathcal{N}$ in Theorem 4.3 we derive the well-known Ganster–Reilly decomposition Theorem 1.1.

References

[1] M.E. Abd El-Monsef, E.F. Lashien and A.A. Nasef, On I-open sets and I-continuous functions, *Kyungpook Math. J.*, **32** (1) (1992), 21-30.

[2] H. Blumberg, New properties of all real functions, *Trans. Amer. Math. Soc.*, **24** (1922) 113-128.

[3] H.H. Corson and E. Michael, Metrizability of certain countable unions, *Illinois J. Math.*, **8** (1964), 351–360.

[4] J. Foran and P. Liebnitz, A characterization of almost resolvable spaces, *Rend. Circ. Mat. Palermo (2)*, **40** (1991), 136–141.

[5] M. Ganster and I.L. Reilly, Locally closed sets and LC-continuous functions, *Internat. J. Math. Math. Sci.*, **3** (1989), 417–424.

[6] M. Ganster and I. Reilly, A decomposition of continuity, *Acta Math. Hungar.*, **56** (3-4) (1990), 299–301.

[7] M. Ganster and I. Reilly, On a decomposition of continuity, *General Topology and Applications*, Vol. **134** Lecture Notes Pure Appl. Math., Dekker, New York, 1991.

[8] M. Ganster and I. Reilly, Another decomposition of continuity, *Annals of the New York Academy of Sciences*, Vol. **704** (1993), 135–141.

[9] E. Hayashi, Topologies defined by local properties, *Math. Ann.*, **156** (1964), 205-215.

[10] D. Janković and T.R. Hamlett, New topologies from old via ideals, *Amer. Math. Monthly*, **97** (1990), 295-310.

[11] D. Janković and T.R. Hamlett, Compatible Extensions of Ideals, *Boll. Un. Mat. It.*, **7** (1992), 453-465.

[12] E.H. Kronheimer, The topology of digital images, *Topology Appl.*, **46** (3) (1992), 279–303.

[13] K. Kuratowski, Topology, Vol. I, Academic Press, New York, 1966.

[14] R.A. Mahmound and D.A. Rose, A note on spaces via dense sets, *Tamkang J. Math.*, **24** (3) (1993), 333–339.

[15] O. Njåstad, On some classes of nearly open sets, *Pacific J. Math.*, **15** (1965), 961-970.

[16] T. Noiri, Hyperconnectedness and pre-open sets, *Rev. Roumaine Math. Pures Appl.*, **29** (4) (1984), 329-334.

[17] P. Samuels, A topology formed from a given topological space, *J. London Math. Soc.*, **10** (2) (1975), 409–416.

[18] J. Tong, A decomposition of continuity, *Acta Math. Hungar.*, **48** (1-2) (1986), 11–15.

[19] R. Vaidyanathaswamy, *Proc. Indian Acad. Sci.*, **20** (1945), 51–61.

E-mail: `dontchev@cc.helsinki.fi`, `dontchev@e-math.ams.org`